Lecture 3
Mantle melting I
Review:

1. Basic Earth Structure.
Earth’s Present Structure

- Core: ~3485 km
- Lithosphere (crust and uppermost solid mantle)
- Mantle
- Asthenosphere
- Crust 0-50 km thick
- Outer core
- Inner core
- Liquid
- Solid

To scale

r ~ 6371 km
Core ~ 3485 km
Review:

1. Basic Earth Structure.
2. Conservation of Energy (Thermal Equation) for the general case.
Last time we introduced the general heat transfer equation -
Now we will consider in more detail the physical significance
of the individual terms. (This is in 2D form).

\[
\frac{\partial \rho e}{\partial t} + \frac{\partial \rho u e}{\partial x} + \frac{\partial \rho v e}{\partial y} = -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} + S - P \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \mu \Phi.
\]

Internal sources/sinks of thermal energy: radioactive decay, latent heat during phase change, etc.

Energy change related to stretching or shearing of fluid.

See online notes from 1/5/09 if interested in derivation of this equation.
Review:

1. Basic Earth Structure.
2. Conservation of Energy (Thermal Equation) for the general case.
Lithosphere:

\[ 0 = k \frac{\partial^2 T}{\partial y^2} + \rho H \]

Here \( H \) are the sum of the radiogenic heat production (product of the heat production of each radiogenic nuclide and its concentration). In the earth this is a function of depth.
Review:

1. Basic Earth Structure.
2. Conservation of Energy (Thermal Equation) for the general case.
4. Conservation of mass (Continuity) for the general case.
Continuity (General case)

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0
\]
Review:

1. Basic Earth Structure.
2. Conservation of Energy (Thermal Equation) for the general case.
4. Conservation of mass (Continuity) for the general case.
Continuity (incompressible case)

\[ \frac{\partial u_i}{\partial x_i} = 0 \]

1D: \[ \frac{\partial u}{\partial x} = 0 \]

2D: \[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

3D: \[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]
Review:

1. Basic Earth Structure.
2. Conservation of Energy (Thermal Equation) for the general case.
4. Conservation of mass (Continuity) for the general case.
6. Heating of the Earth:
   a. radioactive decay
   b. gravitational potential energy
7. Started looking at phase diagrams.
Phase Diagrams:

A phase is any physically distinct and homogeneous (in terms of composition and/or structure) portion that can in theory be separated from other phases.

Phases in a peridotite: olivine, clinopyroxene, orthopyroxene, spinel.
Components:

Minimum number of chemical formulae needed to describe each phase.

Olivine, for example can be described as a solid solution mixture of two components, $\text{Fe}_2\text{SiO}_4$ and $\text{Mg}_2\text{SiO}_4$. 
Gibbs Phase Rule:

\[ F = C - \phi + 2 \]

Number of intensive variables (Temperature and Pressure).
Unary systems (one component):
Two component systems (binary).
Binary systems (eutectic).

![Diagram showing a binary system with phase changes and compositional changes between Diopside and Anorthite, with specific temperatures and compositions marked.](image)
Real rocks are more complex.... But the bulk can be summarized as below for peridotite.
A quick note on pressure:

Lithostatic pressure is equal to the force exerted by gravity of the column of mass above it.

\[ P = \int \rho g dy \]

If the pressure and temperature is known, the proceeding graph can be used to determine if melt is present.
During lecture 2 we discussed solutions to the heat transfer equation for steady conditions in the lithosphere, far from plate boundaries.

But what about the asthenosphere?

The asthenosphere convects by solid-state creep.

To treat this convection in detail we will need to further consider the momentum equation - but for now we will consider some simple scaling.
Convection occurs because most materials expand when they get hotter.

Volumetric coefficient of thermal expansion

\[ \rho = \rho_0 \alpha_v (T - T_0) \]
Critical Rayleigh Number - a criteria for convection.

1. When a buoyant layer underlies a denser layer, it is unstable. However, for the parcel of fluid to convect in a viscous medium, it must overcome viscous forces.

2. As long as a parcel is warm, buoyancy will be important.

\[ \frac{1}{t} \propto \frac{\kappa}{L^2} \]

3. If we substitute for \( t \) using the rise time \( L/U \) then:

\[ \frac{U}{L} \propto \frac{\kappa}{L^2} \]
4. We can make some estimates of the velocity by considering a simplified balance of forces…

Buoyancy balanced by viscous drag.

5. Buoyancy:

\[
\approx (\rho - \rho_0)gL^3 \\
\approx \left( \rho_0 + \rho_0 \alpha_v (T - T_0) \right) - \rho_0 \right)gL^3 \\
\approx (\rho_0 \alpha_v (T - T_0))gL^3
\]
6. Drag force (in this viscous situation) is \( \sim \) shear \( \times \) surface area…

\[
\approx \mu \left[ \frac{U}{L} \right] \times L^2
\]

7. For drag and buoyancy to counteract each other:

\[
U \approx \frac{\rho_0 \alpha_v (T - T_0) g L^2}{\mu}
\]
8. If we now ratio the buoyant and viscous terms, we get:

\[ \sim \frac{\rho_0 \alpha_v (T - T_0) g L^3}{\kappa \mu} \]

9. This balance is called a Rayleigh number.

10. We will be revisiting this concept with more rigor later on - but at this point note that if Ra>657, or the critical Rayleigh number for this simplified case, convection will occur.
To extend the geotherm, let’s make the assumption that because the asthenosphere has a relatively large Peclet number, let’s assume no conductive heat loss.

A parcel of fluid will change temperature slightly during advection due to work on the surrounding environment (or vice versa). This type of path is termed adiabatic.

More formally:

$$\left( \frac{\partial T}{\partial P} \right)_s = \frac{\alpha_v T}{\rho c_p}$$

Isentropic - constant entropy.
Recall that pressure varies with depth according to

\[ dP = d(\rho gy) \]

Then if we assume density and gravity are nearly constant with depth:

\[ \frac{\partial T}{\partial y} = \frac{\alpha_v g T}{c_p} \]

Then integrating,

\[ T = T_{LA} \exp \left[ \frac{\alpha_v g (y - H_{LA})}{c_p} \right] \]
In reality some conductive heat loss is important, as is radioactive decay and changes in rheology in the asthenosphere.

The expression below takes into account a power-law rheology and radioactive decay to give the average temperature of the mantle. We won’t derive this at this point, but it is useful to consider how the average temperature of the mantle has changed through time:

\[
\overline{T}_a = \frac{E_a \overline{T}_a^0}{(t - t_p)(3RT_a^0 \lambda) + E_a}
\]

The equivalent expression for the cooling rate is:

\[
\frac{dT}{dt} = -3\lambda \left[ \frac{RT^2}{E_a} \right]
\]
Peridotite summary:

Base of lithosphere:
Classification of volcanic rocks:
To include melting in our heat transfer equation, we need a way to describe the latent heat cost of the phase change.

This is typically done with a melt fraction to temperature relationship.
The latent term can be given by:

\[ R = \rho L \frac{\partial f}{\partial t} \]